Homework Set 6
Physics 319
Classical Mechanics

Problem 11.5

a) The Lagrangian for the problem is

$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = T - U = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{k}{2}x_1^2 - \frac{k}{2}(x_2 - x_1)^2$$

The two equations of motion in matrix form are

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

The secular equation is

$$\det\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} = 0$$

$$\left(2k - m\omega^2\right)\left(k - m\omega^2\right) - k^2 = 0$$

$$\omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0 \qquad \omega_0^2 = k/m$$

$$\omega^2 = \omega_0^2\,\frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

b) The normal mode amplitudes are

$$\begin{pmatrix} 1/2 \mp \sqrt{5}/2 & -1 \\ -1 & -1/2 \mp \sqrt{5}/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$x_2 = \left(1/2 - \sqrt{5}\right)x_1 \rightarrow x_2 = -0.618 x_1 \text{ for + mode}$$

$$x_2 = \left(1/2 + \sqrt{5}\right)x_1 \rightarrow x_2 = +1.618 x_1 \text{ for } - \text{ mode}$$

Problem 11.14

a) The kinetic energy is

$$T = \frac{m}{2}L^2\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right)$$

The potential energy in the small angle approximation is

$$U = -mgL\cos\phi_1 - mgl\cos\phi_2 + \frac{kL^2}{2}(\phi_2 - \phi_1)^2 \approx \frac{mgL}{2}\left(\phi_1^2 + \phi_2^2\right) + \frac{kL^2}{2}(\phi_2 - \phi_1)^2$$

The Lagrangian for the problem is

$$\mathcal{L} = T - U = \frac{m}{2}L^2\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right) + \frac{mgL}{2}\left(\phi_1^2 + \phi_2^2\right) + \frac{kL^2}{2}(\phi_2 - \phi_1)^2$$

The two equations of motion in matrix form are

$$\begin{pmatrix} mL^2 & 0 \\ 0 & mL^2 \end{pmatrix}\begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{pmatrix} + \begin{pmatrix} mgL+kL^2 & -kL^2 \\ -kL^2 & mgL+kL^2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0.$$

b) The secular equation is

$$\det\begin{vmatrix} \omega_g^2+\omega_k^2-\omega^2 & -\omega_k^2 \\ -\omega_k^2 & \omega_g^2+\omega_k^2-\omega^2 \end{vmatrix} = 0$$

$$\left(\omega_g^2+\omega_k^2-\omega^2\right)^2 = \omega_k^2$$

$$\omega^2 = \omega_g^2, \omega_g^2+2\omega_k^2 \qquad \omega_g^2 = g/L, \omega_k^2 = k/m$$

The normal mode amplitudes are

$$\begin{pmatrix} \pm 1 & -1 \\ -1 & \pm 1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0$$

$$\phi_2 = \phi_1 \text{ for } \omega_g^2 \text{ mode}$$

$$\phi_2 = -\phi_1 \text{ for } \omega_g^2{+}2\omega_k^2 \text{ mode}$$

Problem 11.19

a) In this problem there is a bit of preparation needed because the constraint relates the (moving) cart and pendulum positions. If the cart position is x_1 then the bob position is

$$x_2 = x_1 + L\sin\phi$$
$$y_2 = -mgL\cos\phi.$$

Then

$$\dot{x}_2 = \dot{x}_1 + L\dot{\phi}\cos\phi$$
$$y_2 = L\dot{\phi}\sin\phi.$$

So the Lagrangian for the problem is

$$\mathcal{L}\left(x,\phi,\dot{x},\dot{\phi}\right) = T-U = \frac{m+M}{2}\dot{x}^2 + M\dot{x}L\cos\phi\dot{\phi} + \frac{M}{2}\left(L\dot{\phi}\right)^2 - \frac{k}{2}x^2 + MgL\cos\phi$$

$$\approx \frac{m+M}{2}\dot{x}^2 + M\dot{x}L\dot{\phi} + \frac{M}{2}\left(L\dot{\phi}\right)^2 - \frac{k}{2}x^2 - \frac{MgL}{2}\phi^2$$

The two equations of motion in matrix form are

$$\begin{pmatrix} m+M & ML \\ ML & ML^2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} k & 0 \\ 0 & MgL \end{pmatrix}\begin{pmatrix} x \\ \phi \end{pmatrix} = 0.$$

b) For the given parameters the secular equation is

$$\det\begin{vmatrix} 2-2\omega^2 & \omega^2 \\ \omega^2 & 1-\omega^2 \end{vmatrix} = 0$$

$$2-4\omega^2+\omega^4 = 0$$

$$\omega^2 = \frac{4\pm\sqrt{16-8}}{2} = 2\pm\sqrt{2}$$

The normal mode amplitudes are

$$\begin{pmatrix} -2\mp 2\sqrt{2} & 2\pm\sqrt{2} \\ 2\pm\sqrt{2} & -1\mp\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ \phi \end{pmatrix}=0$$

$$\sqrt{2}x=\phi \text{ for + mode}$$

$$-\sqrt{2}x=\phi \text{ for } - \text{ mode}$$

The associations between the mode displacements and frequencies seems to be reversed in Taylor's solution in the back.

Problem 11.28

a) Following the solution process from Problem 11.19 the Lagrangian for the problem is

$$\mathcal{L}\left(x,\phi,\dot{x},\dot{\phi}\right)=T-U=\frac{m+M}{2}\dot{x}^2+M\dot{x}L\cos\phi\dot{\phi}+\frac{M}{2}\left(L\dot{\phi}\right)^2+MgL\cos\phi$$

$$\approx\frac{m+M}{2}\dot{x}^2+M\dot{x}L\dot{\phi}+\frac{M}{2}\left(L\dot{\phi}\right)^2-\frac{MgL}{2}\phi^2$$

The two equations of motion in matrix form are

$$\begin{pmatrix} m+M & ML \\ ML & ML^2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\phi} \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & MgL \end{pmatrix}\begin{pmatrix} x \\ \phi \end{pmatrix}=0.$$

The normal mode equation is

$$\det\begin{vmatrix} -\left(1+m/M\right)\omega^2 & -L\omega^2 \\ -\omega^2/L & \omega_g^2-\omega^2 \end{vmatrix}=0$$

$$-\left(1+m/M\right)\omega^2\left(\omega_g^2-\omega^2\right)-\omega^4=0$$

$$\omega^2=0,\omega_g^2\left(1+M/m\right)$$

b) The "zero" mode is simply that the whole system can be translated with a uniform velocity without any relative motion of the bodies (zero frequency means no motion). For there to remain no motion the system the pendulum bob must be vertical, i.e., $\phi=0$. The non-zero mode has normal mode amplitude

$$\begin{pmatrix} -\left(1+m/M\right)\left(1+M/m\right) & -L\left(1+M/m\right) \\ -\left(1+M/m\right)/L & -M/m \end{pmatrix}\begin{pmatrix} x \\ \phi \end{pmatrix}=0$$

$$x/L=-\phi/\left(1+m/M\right)$$

The oscillation "must" be antisymmetric so that the *x* momentum in the whole system $m\dot{x}_1+M\dot{x}_2=\left(m+M\right)\dot{x}_1+ML\dot{\phi}$ (see 11.19) remains zero when the oscillation is present.

Problem 11.32

a) The Lagrangian for the problem is

$$\mathcal{L}=T-U=\frac{m}{2}\left(\dot{x}_1^2+\dot{x}_3^2\right)+\frac{M}{2}\dot{x}_2^2-\frac{k}{2}\left(x_2-x_1\right)^2-\frac{k}{2}\left(x_3-x_2\right)^2$$

The three equations of motion in matrix form are

$$\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The secular equation is

$$\det\begin{vmatrix} \omega_0^2 - \omega^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 - M\omega^2/m & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0$$

$$\left(\omega_0^2 - \omega^2\right)^2 \left(2\omega_0^2 - M\omega^2/m\right) - 2\omega_0^4\left(\omega_0^2 - \omega^2\right) = 0$$

$$\omega^2 = 0, \omega_0^2, \left(1 + 2m/M\right)\omega_0^2$$

b) and c) The normal mode amplitude eigenvectors are

$$\lambda_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \lambda_{\omega_0} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \lambda_{(1+2m/M)\omega_0} = \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$$

As above, the zero mode corresponds to translation of the whole system without any changes. Such a motion would not induce any oscillation in the springs.